Rotating With Quaternions - The Maths

Duncan J. Adamson

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1 What Is A Quaternion

1.1 Imaginary Numbers

When calculating the square of a number, the answer is always greater than or equal to 0.

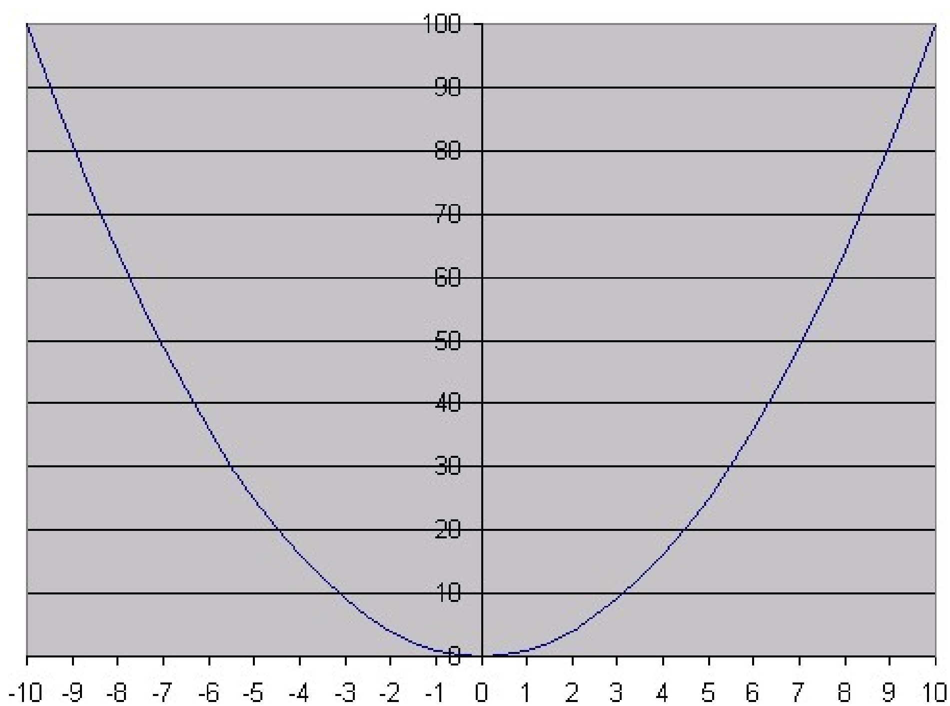

Figure 1: $y = x^2$

This leads to the question, what is $\sqrt{-1}$. As we don't know, we will simply call it i, the unit imaginary componant.

$$i^2 = -1$$

An imaginary number therefore is one with an normal componant and an imaginary component, e.g. $7i$, $2 + 3.6i$ or $-1.1 - 8i$.

Like normal numbers, there are arithmetic operations:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(a + bi) * (c + di) = (a * c - b * d) + (a * d + b * c)i$$

$$(a + bi)/(c + di) = /frac(a + bi) * (c - di)(c + di)(c - di)$$

Each imaginary number q has a conjugate $q*$. This is defined as

$$q = a + bi$$

$$q* = a - bi$$

1.2 Quaternions

Quaternions are extensions of imaginary numbers. Each quaternion has one normal componant and three imaginary componants i, j and k. These are defined such that:

$$i^2 = j^2 = k^2 = -1$$

$$i * j = -j * i = k$$

$$j * k = -k * j = i$$

$$k * i = -i * k = j$$

A quaternion q and its conjugate $q*$ are defined as:

$$q = ai + bj + ck + d$$

$$q* = -ai - bj - ck + d$$

Quaternions are often represented as vectors:

$$q = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Addition and subtraction operaters for quaternions are similar to those for imaginary numbers.

Multiplication is defined as:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \begin{pmatrix} ah + de + bg - cf \\ bh + df + ce - ag \\ ch + dg + af - be \\ dh - ae - bf - cg \end{pmatrix}$$

2 How Do I Rotate With A Quaternion

If we want to rotate a point v around axis A (of unit length) by angle θ, we define two quaternions:

First, the quaternion for the point v that we want to rotate:

$$p = \begin{pmatrix} X_v \\ Y_v \\ Z_v \\ 1 \end{pmatrix}$$

Then the quaternion for the rotation operation:

$$q = \begin{pmatrix} X_A * sin(\frac{\theta}{2}) \\ Y_A * sin(\frac{\theta}{2}) \\ Z_A * sin(\frac{\theta}{2}) \\ cos(\frac{\theta}{2}) \end{pmatrix}$$

The result of the rotation, r can be found by simply calculating the following quaternion multiplication:

$$r = qpq*$$

3 Example

Given v, A and θ,

$$v = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{1}{sqrt(41)} \\ \frac{-2}{sqrt(41)} \\ \frac{6}{sqrt(41)} \end{pmatrix}$$

$$\theta = 30$$

We can prepare p and q,

$$p = \begin{pmatrix} 3 \\ 7 \\ -2 \\ 1 \end{pmatrix}$$

$$q = \begin{pmatrix} \frac{sin15}{sqrt(41)} \\ \frac{-2*sin(15)}{sqrt(41)} \\ \frac{6*sin(15)}{sqrt(41)} \\ cos(15) \end{pmatrix}$$

And finally calculate r.

$$r = \begin{pmatrix} \frac{sin15}{sqrt(41)} \\ \frac{-2*sin(15)}{sqrt(41)} \\ \frac{6*sin(15)}{sqrt(41)} \\ cos(15) \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ -2 \\ 1 \end{pmatrix} \begin{pmatrix} \frac{-sin15}{sqrt(41)} \\ \frac{2*sin(15)}{sqrt(41)} \\ \frac{-6*sin(15)}{sqrt(41)} \\ cos(15) \end{pmatrix} = \begin{pmatrix} -0.44438 \\ 7.774228 \\ -1.16786 \\ 1 \end{pmatrix}$$